DSC 190

Data Structures & Algorithms

Lecture 10 | Part 1

Today's Lecture

Beyond Greedy

- Greedy algorithms are typically **fast**, but may not find the optimal answer.
- Brute force guarantees the optimal answer, but is **slow**.
- Can we guarantee the optimal answer and be faster than brute force?

Today

- The **backtracking** idea.
- It is a useful, general algorithm design technique[1].
- And the foundation of **dynamic programming**.

[1] Commonly seen in tech interviews

DSC 190

Data Structures & Algorithms

Lecture 10 | Part 2

The 0-1 Knapsack Problem

0-1 Knapsack

- Suppose you're a thief.
- You have a knapsack (bag) that can fit 100L.
- And a list of n things to possibly steal.

item	size (L)	price
TV	50	\$400
iPad	2	\$900
Printer	10	\$100
⋮	⋮	⋮

- Goal: maximize total value of items you can fit in your knapsack.

Example

item	size (L)	price
1	50	$40
2	10	$25
~~3~~	~~80~~	~~$100~~
4	5	$10
~~5~~	~~20~~	~~$20~~
6	30	$6
7	8	$32
8	17	$34

In the bag: 3, 5

Total value: $120

Space remaining: 0

Greedy

- Does a greedy approach find the optimal?
- What do we mean by “greedy”?
- Idea #1: take most expensive available that will fit.

Example

item	size (L)	price
~~1~~	~~50~~	~~$40~~
~~2~~	~~10~~	~~$25~~
3	80	$100
~~4~~	~~5~~	~~$10~~
~~5~~	~~20~~	~~$20~~
~~6~~	~~30~~	~~$6~~
~~7~~	~~8~~	~~$32~~
8	17	$34

In the bag: 3, 8

Total value: $134

Space remaining: 3

Greedy, Idea #2

- We want items with high value for their size.
- Define "price density" =
  `item.price / item.size`
- Idea #2: take item with highest price density.

Example

90

32
25
10
34
20
6

item	size (L)	price	PD
~~1~~	~~50~~	~~$40~~	~~0.80~~
~~2~~	~~10~~	~~$25~~	~~2.50~~
~~3~~	~~80~~	~~$100~~	~~1.25~~
~~4~~	~~5~~	~~$10~~	~~2.00~~
~~5~~	~~20~~	~~$20~~	~~1.00~~
~~6~~	~~30~~	~~$6~~	~~0.20~~
~~7~~	~~8~~	~~$32~~	~~4.00~~
8	17	$34	2.00

In the bag: 7, 2, 4, 8, 5, 6

Total value: $127

Space remaining: 10

Greedy is Not Optimal

- Claim: the best possible total value is \$157.
  - Items 2, 3, and 7.

Never Looking Back

- Once greedy makes a decision, it never looks back.
- This is why it may be suboptimal.
- **Backtracking**: go back to reconsider every previous decision.

DSC 190

Data Structures & Algorithms

Lecture 10 | Part 3

Backtracking

Backtracking

- Reconsider every decision.
- If we initially tried including x, also try *not* including x.
    - Find the best solution among those that **include** x
    - Find the best solution among those that **exclude** x
    - Return the better of the two.

Backtracking

```python
def knapsack(items, bag_size):
    # choose item arbitrarily from those that fit in bag
    x = items.arbitrary_item(fitting_in=bag_size)

    # if None, it means there was no item that fit
    if x is None:
        return 0

    # assume x should be in bag, see what we get
    best_with = ...

    # backtrack: now assume x should not be in bag, see what we get
    best_without = ...

    return max(best_with, best_without)
```

Recursive Subproblems

- What is BEST(`items, bag_size`) if we assume that `x` **is** in the bag?
- The best outcome is `x.price` + best choice of remaining items.
- Imagine choosing `x`.
  - Your current total value is `x.price`.
  - You have `bag_size - x.size` space left.
  - Items left to choose from: `items - x`.
- Answer: `x.price` + BEST(`items - x, bag_size - x.size`)

Recursive Subproblems

- What is BEST(`items, bag_size`) if we assume that `x` **is not** the bag?
- Clearly, you want the best outcome for remaining items.
- Imagine deciding `x` is not in the bag.
  - Your current total value is `0`.
  - You have `bag_size` space left.
  - Items left to choose from: `items - x`.
- Answer: `0` + BEST(`items - x, bag_size`)

Backtracking

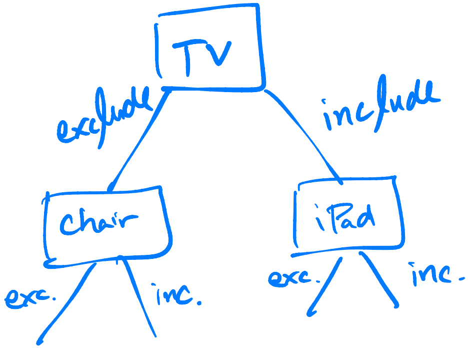
```python
def knapsack(items, bag_size):
    # choose item arbitrarily from those that fit in bag
    x = items.arbitrary_item(fitting_in=bag_size)

    # if None, it means there was no item that fit
    if x is None:
        return 0

    # assume x is in the bag, see what we get
    best_with = x.price + knapsack(items - x, bag_size - x.size)

    # now assume x is not in bag, see what we get
    best_without = 0 + knapsack(items - x, bag_size)

    return max(best_with, best_without)
```

Backtracking

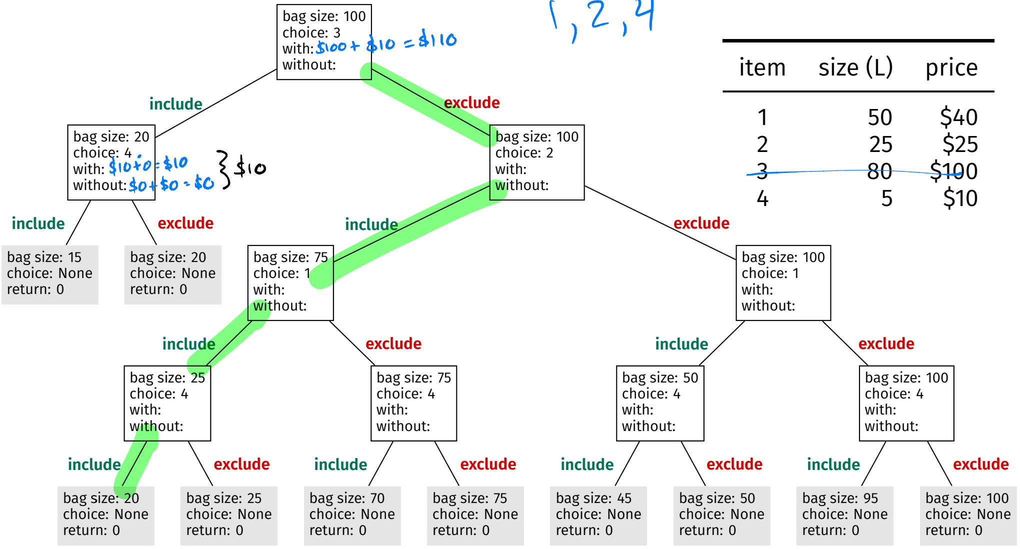
```python
def knapsack(items, bag_size):
    # choose item arbitrarily from those that fit in bag
    x = items.arbitrary_item(fitting_in=bag_size)

    # if None, it means there was no item that fit
    if x is None:
        return 0

    items.remove(x)
    best_with = x.price + knapsack(items, bag_size - x.size)
    best_without = knapsack(items, bag_size)
    items.replace(x)

    return max(best_with, best_without)
```

Backtracking

- **Backtracking**: go back to reconsider every previous decision.
- Searches the whole tree.
- Can be thought of as a DFS on implicit tree.



1, 2, 4



item	size (L)	price
1	50	\$40
2	25	\$25
~~3~~	~~80~~	~~\$100~~
4	5	\$10

Exercise

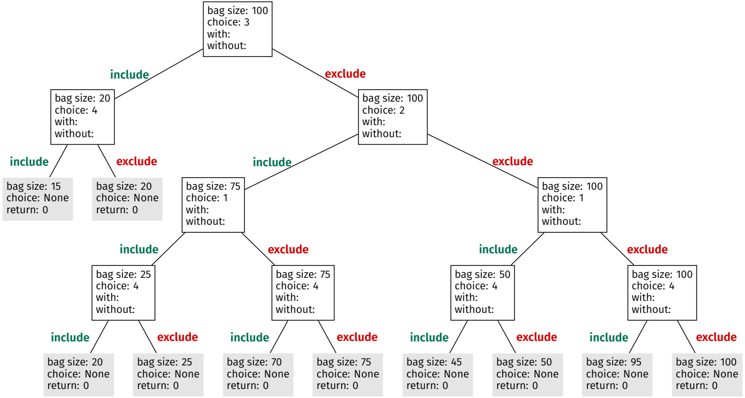
Is the backtracking solution **guaranteed** to find an optimal solution?

Yes!

- It tries every **valid** combination and keeps the best.
  - A combination of items is valid if they fit in the bag together.

Leaf Nodes

- Each leaf node is a different valid combination.



Exercise

Suppose instead of choosing an **arbitrary** node we choose most **expensive**. Is it still highlight guaranteed to find an optimal solution?

Yes!

- The choice of node is **arbitrary**.
- Call tree will change, but all valid combinations are still tried.

Exercise

How does backtracking relate to the greedy approach? How would you change the code to make it greedy?

Summary

```python
def knapsack_greedy(items, bag_size):
    # choose greedily
    x = items.most_valuable_item(fitting_in=bag_size)

    # if None, it means there was no item that fit
    if x is None:
        return 0

    # assume x is in the bag, see what we get
    best_with = x.price + knapsack(items - x, bag_size - x.size)

    # in the greedy approach, we don't do this
    # best_without = knapsack(items - x, bag_size)

    return best_with
```

DSC 190

Data Structures & Algorithms

Lecture 10 | Part 4

Efficiency Analysis

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$$

A Benchmark

- Brute force: try every **possible** combination of items.
  - Even the **invalid** ones whose total size is too big.
  - Why? Hard to know which are invalid without trying them.
- There are $\Theta(2^n)$ possible combinations.
- So brute force takes $\Omega(2^n)$ time. **Exponential :(**

$T(n) = 2T(n/2) + n = \Theta(n \lg n)$

Time Complexity of Backtracking

$T(n-1) = 2T(n-2) + 1$

```python
def knapsack(items, bag_size):
    # choose item arbitrarily from those that fit in bag
    x = items.arbitrary_item(fitting_in=bag_size)

    # if None, it means there was no item that fit
    if x is None:
        return 0

    items.remove(x)
    best_with = x.price + knapsack(items, bag_size - x.size)
    best_without = knapsack(items, bag_size)
    items.replace(x)

    return max(best_with, best_without)
```

$T(n) = 2T(n-1) + 1$

$= 2^k T(n-k) + k$

$k = n$

$= 2^n T(0) + n$

$= \Omega(2^n)$

:(

Backtracking Takes **Exponential Time**

- ...in the worst case.
- This is just as bad as **brute force**.
- So why use it?
- Its worst case isn't always indicative of its practical performance.

Intuition

- Brute force tries all **possible** combinations.
  - E.g., all combinations of items, even if they don't fit in the bag.
- Backtracking tries all **valid** combinations.
  - E.g., all combinations of items that will fit in the bag.
- The number of valid combinations can be **much less** than the number of possible combinations.[2]

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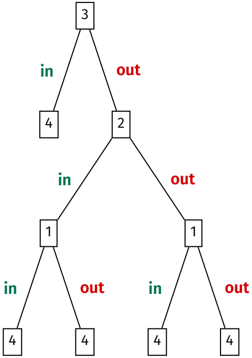
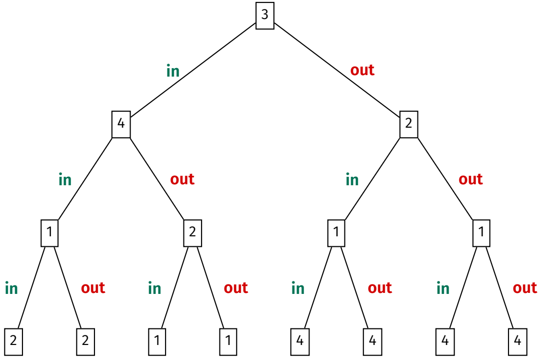
[2]Not always true!

Pruning



backtracking



brute force

Pruning

- Backtracking **prunes** branches that lead to invalid solutions.

Example

- 23 items with size/price chosen from $\mathrm{Unif}([23, \ldots, 46])$
- Bag size is 46
- Brute force: ?
- Backtracking: ?

Example

- 23 items with size/price chosen from Unif([23, ..., 46])
- Bag size is 46
- Brute force: 52 seconds.
- Backtracking: ?

Example

- 23 items with size/price chosen from $\text{Unif}([23, ..., 46])$
- Bag size is 46
- Brute force: 52 seconds.
- Backtracking: 4 milliseconds.

Example

- 300 items with size/price chosen from Unif([150, ..., 300])
- Bag size is 600
- Brute force: ?
- Backtracking: ?

Example

- 300 items with size/price chosen from Unif([150, ..., 300])
- Bag size is 600
- Brute force: $\approx 4.6 \times 10^{77}$ years
- Backtracking: ?

Example

- 300 items with size/price chosen from Unif([150, ..., 300])
- Bag size is 600
- Brute force: $\approx 4.6 \times 10^{77}$ years
- Backtracking: 30 seconds.

Exercise

What is the **worst possible situation** for backtracking? That is, when can we **not** prune any branches?

Backtracking Worst Case

- knapsack's **worst case** is when **bag size is very large**.
- All solutions are valid, aren't pruned.
- But this is actually an easy case!

Exercise

What is the optimal solution when the bag is very large (i.e., can fit everything)?

```python
def knapsack_2(items, bag_size):
    if sum(item.size for item in items) < bag_size:
        return sum(item.price for item in items)

    x = items.arbitrary_item(fitting_in=bag_size)

    if x is None:
        return 0

    items.remove(item)
    best_with = x.price + knapsack_2(items, bag_size - x.size)
    best_without = knapsack_2(items, bag_size)
    items.replace(x)

    return max(best_with, best_without)
```

Pruning

- This further prunes the tree, resulting in speedup for some inputs.

DSC 190

DATA STRUCTURES & ALGORITHMS

Lecture 10 | Part 5

Branch and Bound

Example

- Suppose you have a bag of size 100.
- One of the items is a diamond.
  - Price: $10,000. Size: 1
- The other 49 items are coal.
  - Price: $1. Size: 1
- Do you even consider not taking the diamond?

Idea

1. Assume we take the diamond, compute best result.
2. Find quick upper bound for not taking diamond.
3. If upper bound is less than best for diamond, don't go down that branch.

- This is **branch and bound**; another way to prune tree.

Branch and Bound

```
def knapsack_bb(items, bag_size, find_upper_bound):
    # try to make a good first choice
    x = items.item_with_highest_price_density(fitting_in=bag_size)

    if x is None:
        return 0

    items.remove(item)
    best_with = x.price + knapsack_bb(items, bag_size - x.size)

    upper_bound_without = find_upper_bound(items, bag_size)
    if  upper_bound_without > best_with:
        # we have to look down the other branch...
        best_without = knapsack_bb(items, bag_size)
    else:
        # prune that branch; don't look down it
        best_without = 0

    items.replace(x)

    return max(best_with, best_without)
```

A Good First Choice

- Before, the first choice didn't affect efficiency.
  - We still explored all valid options.
- Now, it does.
  - A good first choice allows us to prune more branches.

Example

item	size (L)	price
1	50	$40
2	25	$25
3	95	$1000
4	5	$10

3

include

exclude

< $100

$1010

Upper Bounds for Knapsack

- How do we get a good upper bound?
- One idea: the solution to the *fractional* knapsack problem upper bounds that for 0/1 knapsack.

DSC 190

Data Structures & Algorithms

Lecture 10 | Part 6

Summary

Summary

- A backtracking approach is **guaranteed** to find an optimal answer.
- It is typically faster than brute force, but can still take **exponential time**.

Generalization

- Backtracking works for a **very wide range** of discrete optimization problems.
- Generalizes beyond “include or exclude” binary decision trees.
  - Any situation where you have a set of choices, and you can only pick one.

Summary

- We can speed up backtracking by pruning:

- Three ways to prune:
    1. Prune invalid branches (default).

    2. Prune “easy” cases.

    3. Prune by branching and bounding.

Summary

- Next time: **dynamic programming**.
- We'll see it is "just" backtracking + a cache.