DSC 190 - Homework 06

Due: Wednesday, May 11

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. Unless otherwise noted by the problem's instructions, show your work or provide some justification for your answer. Homeworks are due via Gradescope at 11:59 PM.

Problem 1.

In lecture, we designed a cost function for the embedding of n points into $\mathbb{R}^1$ using the coordinates of an embedding vector, $\vec{f}$:

$$\text{Cost}(\vec{f}) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(f_i - f_j)^2$$

We then said that this can also be written in the form:

$$\text{Cost}(\vec{f}) = \vec{f}^T L \vec{f},$$

where $L = D - W$ is the (unnormalized) graph Laplacian matrix.

Show that

$$\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(f_i - f_j)^2 = \vec{f}^T L \vec{f}.$$

in the simple setting where $n = 2$, and $\vec{f} = (f_1, f_2)^T$.

You may assume that the weight matrix, W, is symmetric.

Solution:

On the left hand side, we have:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(f_i - f_j)^2 = w_{11}(f_1 - f_1)^2 + w_{12}(f_1 - f_2)^2 + w_{21}(f_2 - f_1)^2 + w_{22}(f_2 - f_2)^2$$

The first and last terms are zero, since $f_i - f_i = 0$:

$$= w_{12}(f_1 - f_2)^2 + w_{21}(f_2 - f_1)^2$$

Since W is symmetric, we have:

$$\begin{aligned} &= w_{12}(f_1 - f_2)^2 + w_{12}(f_2 - f_1)^2 \\ &= 2w_{12}(f_1 - f_2)^2 \end{aligned}$$

On the right hand side, we have

$$\begin{aligned}
\vec{f}^T L \vec{f} &= \begin{pmatrix} f_1 & f_2 \end{pmatrix} \begin{pmatrix} d_1 - w_{11} & -w_{12} \\ w_{21} & d_2 - w_{22} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\
&= \begin{pmatrix} f_1 & f_2 \end{pmatrix} \begin{pmatrix} f_1(d_1 - w_{11}) - f_2 w_{12} \\ -f_1 w_{21} + f_2(d_2 - w_{22}) \end{pmatrix} \\
&= f_1^2(d_1 - w_{11}) - f_1 f_2 w_{12} - f_2 f_1 w_{21} + f_2^2(d_2 - w_{22})
\end{aligned}$$

Now we need to think a little. First, $w_{12} = w_{21}$ since W is symmetric. This allows us to combine the middle terms.

$$= f_1^2(d_1 - w_{11}) - 2 f_1 f_2 w_{12} + f_2^2(d_2 - w_{22})$$

Next, we somehow need to get rid of the d_1 and d_2, which are the degrees of node 1 and 2 respectively. But remember that d_1 is defined to be $d_1 = w_{11} + w_{12}$, so $d_1 - w_{11} = w_{12}$. Similarly, $d_2 - w_{22} = w_{21} = w_{12}$. Therefore:

$$\begin{aligned}
&= f_1^2 w_{12} - 2 f_1 f_2 w_{12} + f_2^2 w_{12} \\
&= w_{12}(f_1^2 - 2 f_1 f_2 + f_2^2) \\
&= w_{12}(f_1 - f_2)^2
\end{aligned}$$

Note that this is equal to half of $\sum \sum w_{ij}(f_i - f_j)^2$, as shown above.

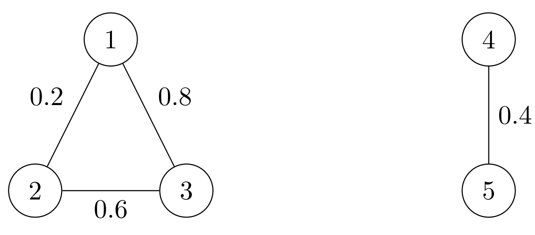
Problem 2.

In lecture, we saw that one minimizer of the cost function

$$\text{Cost}(\vec{f}) = \sum_i \sum_j w_{ij}(f_i - f_j)^2$$

is the vector $\vec{f} = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$ which embeds all of the nodes of the graph to exactly the same number. The cost of this trivial embedding is zero, and we typically ignore it in favor of the eigenvector of the graph Laplacian with the *next* smallest positive eigenvalue for the embedding.

In the case where the graph has multiple connected components, however, there may be additional embeddings which have a cost of zero. For example, consider the similarity graph G shown below:



The weight of each edge is shown; the weight of non-existing edges is zero.

Find a normalized embedding vector $\vec{g}$ for the above graph such that $\text{Cost}(\vec{g}) = 0$ and $\vec{g} \perp \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$; that is, $\vec{g}$ is orthogonal to the vector of all ones.

Solution:

For a connected graph, we can achieve a cost of zero by embedding each node to the exact same place. For a disconnected graph, we can achieve a cost of zero by embedding all nodes *in the same connected* *component* to the same place.

Let $\vec{g} = (g_1, g_2, g_3, g_4, g_5)^T$. In this case, we have a connected component of three nodes and another of two nodes. We can embed each of the three nodes in the first component to the same number, a, and the two nodes of the second component to the same number, b. That is: $\vec{g} = (a, a, a, b, b)^T$.

There are two constraints: $\vec{g}$ is orthogonal to the vector of all ones, and $\|\vec{g}\| = 1$. The latter tells us that $3a^2 + 2b^2 = 1$, and the former tells us that $\vec{g} \cdot (1, 1, 1, 1, 1)^T = 3a + 2b = 0$. Solving for b, we find $b = -3a/2$. Substituting into the first equation:

$$1 = 3a^2 + 2b^2 \implies 1 = 3a^2 + 18a^2/4 \implies 30a^2/4 = 1 \implies a = \sqrt{2/15}$$

And therefore

$$b = -3a/2 = -\frac{3}{2}\sqrt{2/15} = -\sqrt{18/60} = -\sqrt{3/10}$$

So

$$\vec{g} = \begin{pmatrix} \sqrt{2/15} \\ \sqrt{2/15} \\ \sqrt{2/15} \\ -\sqrt{3/10} \\ -\sqrt{3/10} \end{pmatrix}$$

Checking that this is orthogonal to the vector of all ones:

$$3a + 2b = 3\sqrt{2/15} - 2\sqrt{3/10} = \sqrt{18/15} - \sqrt{12/10} = \sqrt{6/5} - \sqrt{6/5} = 0.$$

Checking that this is normalized:

$$3a^2 + 2b^2 = 3 \times (2/15) + 2 \times (3/10) = 6/15 + 6/10 = 2/5 + 3/5 = 1$$