DSC 190

DATA STRUCTURES & ALGORITHMS

Lecture 12 | Part 1

Today's Lecture

Dynamic Programming

- We’ve seen that dynamic programming can lead to fast algorithms that find the optimal answer.
- Today, we’ll see one data science application: longest common substring.
- Used to match DNA sequences, fuzzy string comparison, etc.

The Strategy

1. Backtracking solution.

2. A “nice” backtracking solution with overlapping subproblems.

3. Memoization.

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DATA STRUCTURES & ALGORITHMS

Lecture 12 | Part 2

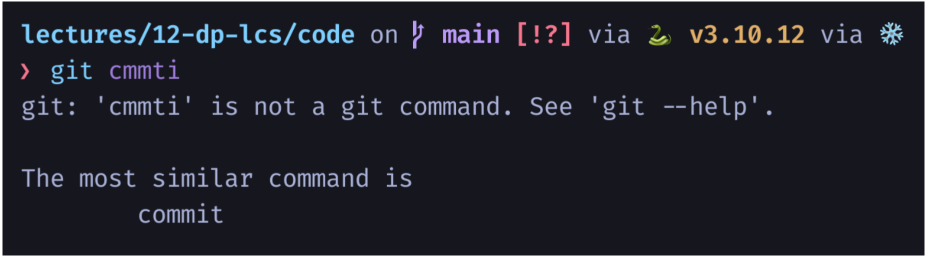
Longest Common Subsequence

Fuzzy String Matching

- Suppose you're doing a sentiment analysis of tweets.
- How do people feel about the University of California?
- Search for: `university of california`
- People can't spell: `uivesity of califrbia`
- How do we recognize the match?

DNA String Matching

- Suppose you're analyzing a genome.
- DNA is a sequence of `G,A,T,C`.
- Mutations cause same gene to have slight differences.
- Person 1: `GATTACAGATTACA`
- Person 2: `GATCACAGTTGCA`



```
lectures/12-dp-lcs/code on  main [!?] via 🐍 v3.10.12 via ❄️
❯ git cmmti
git: 'cmmti' is not a git command. See 'git --help'.

The most similar command is
        commit
```

Measuring Differences

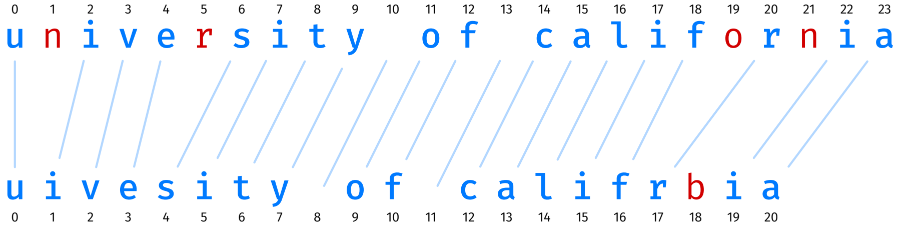
- Given two strings of (possibly) different lengths.
- Measure how similar they are.
- One approach: **longest common subsequences**.

Common Subsequences

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
u	n	i	v	e	r	s	i	t	y		o	f		c	a	l	i	f	o	r	n	i	a

u	i	v	e	s	i	t	y		o	f		c	a	l	i	f	r	b	i	a
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Common Subsequences



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
u	n	i	v	e	r	s	i	t	y		o	f		c	a	l	i	f	o	r	n	i	a

u	i	v	e	s	i	t	y		o	f		c	a	l	i	f	r	b	i	a
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Longest Common Subsequences

- We will measure similarity by finding length of the **longest common subsequence** (LCS).
- Now: let’s define the LCS..

Subsequences

s a n d i e g o

s a n d **i** e **g** **o** → **igo**

s a n d **i** e g **o** → **sio**

s **a** n **d** i **e** **g** **o** → **sadego**

s **a** **n** **d** **i** **e** **g** **o** → **sandiego**

Not Subsequences

s a n d i e g o

s a n d i e g o ↛ sea

0 5 1

s a n d i e g o ↛ sooo

0 1 2 3 4 5 6 7

0 7 7 7

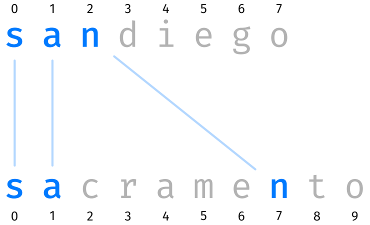
Subsequences

- A **subsequence** of a string `s` of length n is determined by a strictly monotonically increasing sequence of indices with values in $\{0, 1, \ldots, n-1\}$.

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \texttt{s} & \texttt{a} & \texttt{n} & \texttt{d} & \texttt{i} & \texttt{e} & \texttt{g} & \texttt{o} \end{array} \rightarrow \begin{array}{cccccc} 0 & 1 & 3 & 5 & 6 & 7 \\ \texttt{s} & \texttt{a} & \texttt{d} & \texttt{e} & \texttt{g} & \texttt{o} \end{array}$$

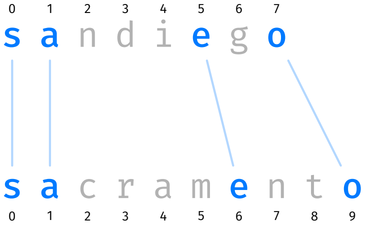
Common Subsequences

- Given two strings, a **common subsequence** is subsequence that appears in both.



Common Subsequences

- Given two strings, a **common subsequence** is subsequence that appears in both.



0 1 2 3 4 5 6 7

s a n d i e g o

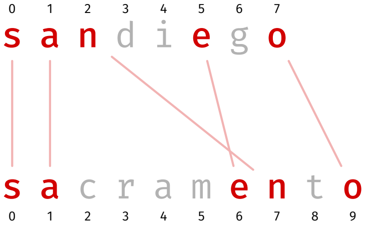
s a c r a m e n t o

0 1 2 3 4 5 6 7 8 9

Not Common Subsequences

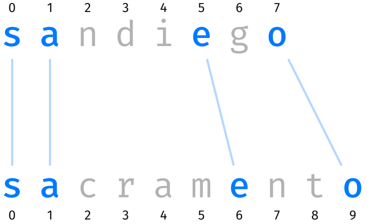
- The lines cannot overlap.

saneo



Longest Common Subsequences

- A **longest common subsequence** (LCS) between two strings is a common subsequence that has the greatest length out of all common subsequences.



0 1 2 3 4 5 6 7

s a n d i e g o

s a c r a m e n t o

0 1 2 3 4 5 6 7 8 9

Main Idea

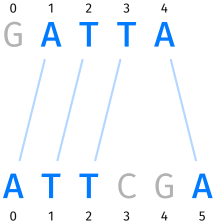
The longer the LCS, the “more similar” the two strings.

Common Subsequences, Formally

- Our backtracking solution will build a common subsequence piece by piece.

- How can we represent the idea of “lines between letters” more formally?

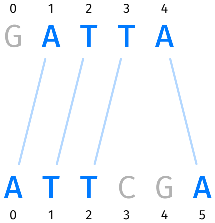
Matching



(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
[✓ (1,0)]	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
(2,0)	[✓ (2,1)]	(2,2)	(2,3)	(2,4)	(2,5)
(3,0)	(3,1)	[✓ (3,2)]	(3,3)	(3,4)	(3,5)
(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	[✓ (4,5)]

Matching

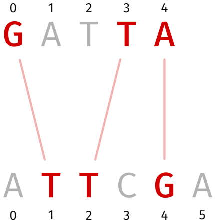
- A **matching** between strings a and b is a set of (i, j) pairs.
- Each (i, j) pair is interpreted as "`a[i]` is paired with `b[j]`".
- Example: $\{(1, 0), (2, 1), (3, 2), (4, 5)\}$



Invalid Matchings

- Not all matchings represent common subsequences!

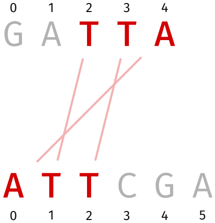
- Example: $\{(0,1),(3,2),(4,4)\}$:



Invalid Matchings

- Not all matchings represent common subsequences!

- Example: $\{(4, 0), (2, 1), (3, 2)\}$:



Valid Matchings

- We'll say a matching M is **valid** if:
  - `a[i] == b[j]` for every pair (i, j); and
  - there are no "crossed lines"

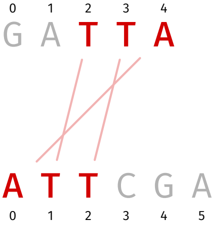
“Crossed Lines”

(1, 4)
(3, 5)

- Suppose (i, j) and (i', j') are in the matching.

- “Crossed lines” occur when either:
  - $i < i'$ but $j \geq j'$; or
  - $i > i'$ but $j \leq j'$.

$i \quad j$
$(4, 0)$
$(2, 1)$
$i' \quad j'$

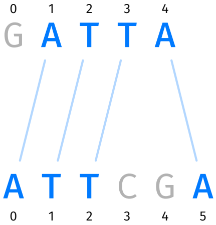


Valid Matchings

- We’ll say a matching M is **valid** if:
  - `a[i] == b[j]` for every pair (i, j); and
  - there are no “crossed lines”. that is, for every choice of distinct pairs $(i, j), (i', j') \in M$:

$$i < i' \text{ and } j < j' \qquad \text{or} \qquad i > i' \text{ and } j > j'$$

- Example: $\{(1, 0), (2, 1), (3, 2), (4, 5)\}$



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DATA STRUCTURES & ALGORITHMS

Lecture 12 | Part 3

Step 01: Backtracking

Road to Dynamic Programming

- We’ll follow same road to a DP solution as last time.

- **Step 01: Backtracking solution.**

- Step 02: A “nice” backtracking solution with overlapping subproblems.

- Step 03: Memoization.

Backtracking

- We'll build up a matching, one pair at a time.
- Choose an arbitrary pair, `(i, j)`.
  - Recursively see what happens if we **do** include `(i, j)`.
  - Recursively see what happens if we **don't** include `(i, j)`.
- This will try **all valid matchings**, keep the best.

Backtracking

[(0,0), (0,1), (0,2)....

```python
def lcs_bt(a, b, pairs):
    """Solve find best matching using the pairs in `pairs`."""
    pair = pairs.arbitrary_pair()

    if pair is None:
        return 0

    i, j = pair

    # best with
    best_with = ...

    # best without
    best_without = ...

    return max(best_with, best_without)
```

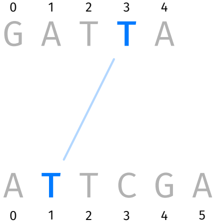
0 1 2 3 4
hello
| (0,2)
helo
0 1 2 3

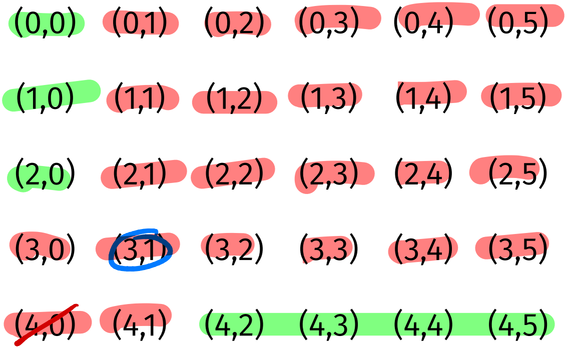
Recursive Subproblems

What is BEST(a, b, pairs) if we assume that (i, j) **is** in matching?

- If a[i] != ~~a~~b[j]:
  - Your current common substring is **invalid**. Length is zero.
  - Don't build matching further.
- If a[i] == ~~a~~b[j]:
  - Your current common substring has length one.
  - Pairs remaining to choose from: those **compatible** with (i,j).
  - You find yourself in a similar situation as before.
  - Answer: 1 + BEST(~~activities~~ pairs.compatible_with(x))

`pairs.compatible_with(x)`





(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
~~(4,0)~~	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)

Backtracking

```python
def lcs_bt(a, b, pairs):
    """Solve find best matching using the pairs in `pairs`."""
    pair = pairs.arbitrary_pair()

    if pair is None:
        return 0

    i, j = pair

    # best with
    if a[i] == b[j]:
        best_with = 1 + lcs_bt(a, b, pairs.compatible_with(i, j))
    else:
        best_with = 0

    # best without
    best_without = ...

    return max(best_with, best_without)
```

Recursive Subproblems

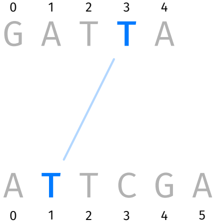
- What is BEST(`a, b, pairs`) if we assume that `(i, j)` **is not** in matching?

- Imagine not choosing `x`.
    - Your current common substring is empty.
    - Activities left to choose from: all except `(i, j)`.

- You find yourself in a similar situation as before.

- Answer: BEST(a, b, `pairs.without(i, j))`)

pairs.without(x)



0 1 2 3 4
G A T T A

A T T C G A
0 1 2 3 4 5

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)

Backtracking

```python
def lcs_bt(a, b, pairs):
    """Solve find best matching using the pairs in `pairs`."""
    pair = pairs.arbitrary_pair()

    if pair is None:
        return 0

    i, j = pair

    # best with
    # assume (i, j) is in the LCS, but only if a[i] == b[j]
    if a[i] != b[j]:
        best_with = 0
    else:
        best_with = 1 + lcs_bt(a, b, pairs.compatible_with(i, j))

    # best without
    best_without = lcs_bt(a, b, pairs.without(i, j))

    return max(best_with, best_without)
```

Backtracking

- This will try all **valid** matchings.
- Guaranteed to find optimal answer.
- But takes exponential time in worst case.

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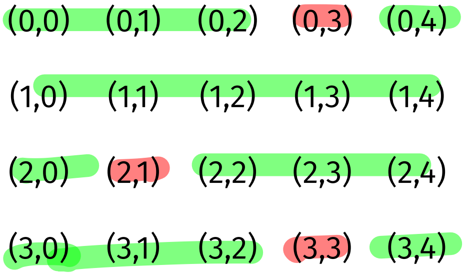
DATA STRUCTURES & ALGORITHMS

Lecture 12 | Part 4

Step 02: A "Nicer" Backtracking Solution

Arbitrary Sets

- In previous backtracking solution, subproblems are arbitrary sets of pairs.
- Rarely see the same subproblem twice.
- This is not good for memoization!



(0,0) (0,1) (0,2) (0,3) (0,4)

(1,0) (1,1) (1,2) (1,3) (1,4)

(2,0) (2,1) (2,2) (2,3) (2,4)

(3,0) (3,1) (3,2) (3,3) (3,4)

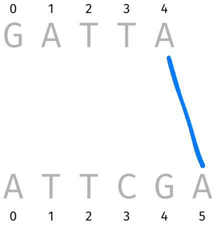
Nicer Subproblems

- In backtracking, we are building a solution piece-by-piece.

- In last lecture, we saw that a careful choice of next piece led to nice subproblems.

- Let’s try choosing the *last* remaining letters from each string as the next piece of the matching.

Last Letters



0 1 2 3 4

G A T T A

A T T C G A

0 1 2 3 4 5

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)

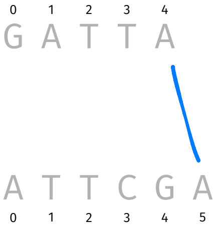
Nicer Backtracking

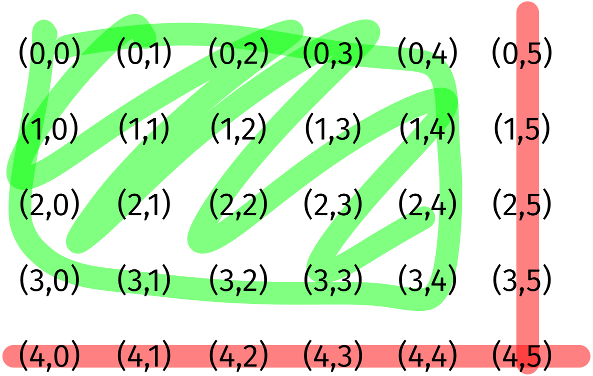
```python
def lcs_bt_nice(a, b, pairs):
    """Solve find best matching using the pairs in `pairs`."""
    pair = pairs.last_pair()

    if pair is None:
        return 0

    i, j = pair

    # best with
    if a[i] != b[j]:
        best_with = 0
    else:
        best_with = 1 + lcs_bt_nice(a, b, pairs.compatible_with(i, j))

    # best without
    best_without = lcs_bt_nice(a, b, pairs.without(i, j))

    return max(best_with, best_without)
```

Subproblems

- There are two subproblems: LCS using `pairs.compatible_with(i, j)` and LCS using `pairs.without(i, j)`
- Are they “nicer”?

pairs.compatible_with(i, j)





(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)

Nicer Subproblems

- By taking (i, j) as bottom-right pair, `pairs.compatible_with(i, j)` is again rectangular.

3, 4

(2, 3)

(1, 3)

- Easily described by its bottom-right pair, $(i - 1, j - 1)$!

(0, 3)

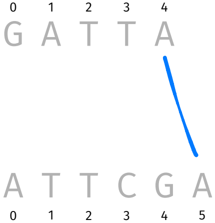
2 , 3

(2, 2)

- Instead of keeping set of `pairs`, just need to pass in i and j of last element.

```python
def lcs_bt_nice_2(a, b, i, j):
    """Solve LCS problem for a[:i], b[:j]."""
    if i < 0 or j < 0:
        return 0

    # best with
    if a[i] != b[j]:
        best_with = 0
    else:
        best_with = 1 + lcs_bt_nice_2(a, b, i-1, j-1)

    # best without
    best_without = ...

    return max(best_with, best_without)
```

pairs.without(i, j)



(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)

Problem

- `pairs.without(i, j)` is **not** rectangular.
- Cannot be described by a single pair.
- But there's a fix.

Observation

- A common substring cannot have pairs both in the last row and the last column. **Crossing lines!**

0	1	2	3	4
G	A	T	T	A

A	T	T	C	G	A
0	1	2	3	4	5

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)

Consequence

- Best(`pairs.without(i, j)`) = max {Best(`pairs.without_row(i)`), Best(`pairs.without_col(j)`)}

0	1	2	3	4
G	A	T	T	A

A	T	T	C	G	A
0	1	2	3	4	5

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)

Observation

- `pairs.without_row(i)` represented by subprob. $(i-1, j)$
- `pairs.without_col(j)` represented by subprob. $(i, j-1)$

0	1	2	3	4
G	A	T	T	A

A	T	T	C	G	A
0	1	2	3	4	5

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)

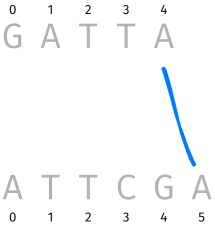
“Nice” Backtracking

```python
def lcs_bt_nice_2(a, b, i, j):
    """Solve LCS problem for a[:i], b[:j]."""
    if i < 0 or j < 0:
        return 0

    # best with
    if a[i] != b[j]:
        best_with = 0
    else:
        best_with = 1 + lcs_bt_nice_2(a, b, i-1, j-1)

    # best without
    best_without = max(
            lcs_bt_nice_2(a, b, i-1, j),
            lcs_bt_nice_2(a, b, i, j-1)
            )

    return max(best_with, best_without)
```

One More Observation

- This is fine, but we can do a little better.

- If `a[i] == b[j]`, we can assume (i, j) is in matching – don’t need to consider otherwise![1]



[1] This is true if we chose last pair; not true if choice was arbitrary.

“Nicer” Backtracking

```python
def lcs_bt_nice_2(a, b, i, j):
    """Solve LCS problem for a[:i], b[:j]."""
    if i < 0 or j < 0:
        return 0

    # best with
    if a[i] == b[j]:
        # best with (i, j)
        return 1 + lcs_bt_nice_2(a, b, i-1, j-1)
    else:
        # best without (i, j)
        return max(
                lcs_bt_nice_2(a, b, i-1, j),
                lcs_bt_nice_2(a, b, i, j-1)
                )
```

Overlapping Subproblems

- Suppose a and b are of length m and n.
- There are mn possible subproblems.
- Backtracking tree has exponentially-many nodes.
- We will see many subproblems over and over again!

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Step 03: Memoization

Backtracking

- The backtracking solutions are slow.
- `a = 'CATCATCATCATCATGAAAAAAAA'`
- `b = 'GATTACAGATTACAGATTACA'`
- “Nice” backtracking solution: 8 seconds.

Backtracking

- The backtracking solutions are slow.
- `a = 'CATCATCATCATCATGAAAAAAAA'`
- `b = 'GATTACAGATTACAGATTACA'`
- “Nice” backtracking solution: 8 seconds.
- Memoized solution: 100 microseconds.

```
def ~~lcs_dp(a, b, i=None, j=None, cache=None):~~
    """Solve LCS problem for a[:i], b[:j]."""
    if i is None:
        i = len(a) - 1

    if j is None:
        j = len(b) - 1

    if cache is None:
        cache = {}

    if i < 0 or j < 0:
        return 0

    if (i,j) in cache:
        return cache[(i, j)]

    # best with
    if a[i] == b[j]:
        # best with (i, j)
        best = 1 + lcs_dp(a, b, i-1, j-1, cache)
    else:
        # best without (i, j)
        best = max(
                lcs_dp(a, b, i-1, j, cache),
                lcs_dp(a, b, i, j-1, cache)
                )

    cache[(i, j)] = best
    return best
```

Top-Down vs. Bottom-Up

- This is the **top-down** dynamic programming solution.
- It takes time $\Theta(mn)$, where m and n are the string lengths.
- To find a bottom-up iterative solution, start with the easiest subproblem.
- What is it?

-1

(3,2)

Bottom-Up Solution

```
# best with
if a[i] == b[j]:
    # best with (i, j)
    best = 1 + lcs_dp(a, b, i-1, j-1, cache)
else:
    # best without (i, j)
    best = max(
            lcs_dp(a, b, i-1, j, cache),
            lcs_dp(a, b, i, j-1, cache)
            )
```

(0,0)	(0,1)	(0,2)
(1,0)	(1,1)	(1,2)
(2,0)	(2,1)	(2,2)
(3,0)	(3,1)	(3,2)

```python
def lcs_dp_bup(a, b):
    """Compute length of LCS, but bottom-up."""
    # initialize cache
    cache = {}
    for i in range(-1, len(a)):
        cache[(i, -1)] = 0
    for j in range(-1, len(b)):
        cache[(-1, j)] = 0

    # fill cache
    for i in range(len(a)):
        for j in range(len(b)):
            if a[i] == b[j]:
                # best with (i, j)
                best = 1 + cache[(i-1, j-1)] # was 1 + lcs_dp(a, b, i-1, j-1, cache)
            else:
                # best without (i, j)
                best = max(
                        cache[(i-1, j)], # was lcs_dp(a, b, i-1, j, cache)
                        cache[(i, j-1)]  # was lcs_dp(a, b, i, j-1, cache)
                        )

            cache[(i, j)] = best

    return cache[(len(a)-1, len(b)-1)]
```

Recoving the Solution

- `lcs_dp` returns the **length** of the LCS.
- How do we recover the actual LCS as a string?
- This information is (implicitly) stored in the cache!

Recovering the Solution

(2,1)

```
a = "ace"
b = "abcde"
```

	-1	0	1	2	3	4
-1	0	0	0	0	0	0
0	0	1	1	1	1	1
1	0	1	1	1	2	2
2	0	1	1	2	2	3

```
# best with
if a[i] == b[j]:
    # best with (i, j)
    best = 1 + lcs_dp(a, b, i-1, j-1, cache)
else:
    # best without (i, j)
    best = max(
            lcs_dp(a, b, i-1, j, cache),
            lcs_dp(a, b, i, j-1, cache)
            )
```

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DATA STRUCTURES & ALGORITHMS

Lecture 12 | Part 6

String Matching in Practice

In Practice

- The **longest common subsequence** is only one way of measuring similarity between strings.

- In fact, LCS is one specific example of an **edit distance**.

Edit Distance

- An **edit** distance is a measure of similarity between two strings.
- It is the minimum number of **edits** required to transform one string into another.
- LCS: only **insert** and **delete** edits allowed.
- **Levenshtein distance**: insert, delete, and **substitute** edits allowed.

In Python

- `difflib` module in the standard library.
- `fuzzywuzzy` module on PyPI.

Next Time

- Find all instances of a **needle** in a **haystack**.