DSC 190 - Homework 05

Due: Monday, November 6

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. Unless otherwise noted by the problem's instructions, show your work or provide some justification for your answer. Homeworks are due via Gradescope at 11:59 PM.

Problem 1.

In lecture, we considered the scheduling problem: given n pairs of event start and finish times, $[s_i, f_i)$, find a schedule containing the greatest number of non-overlapping events. We saw that the greedy strategy of picking events in order of their finish times is guaranteed to be optimal.

a) Given the following start and finish times, find an optimal schedule:

Event	Start	Finish
a	0	2
b	0	1
c	3	4
d	1	3
e	4	6
f	2	5
g	5	7

Solution: Applying the greedy algorithm, we obtain the following schedule (which is guaranteed to be optimal): b, d, c, e.

b) Just because a problem is solved by a greedy solution it does not mean that the greedy solution is the *only* solution.

True or False: in general, an optimal solution to the scheduling problem **must** contain the event with the earliest finish time.

If you answer "False", provide a counterexample in the form of an set of start/finish times and an optimal schedule that does not contain the event with the earliest finish time. If you answer "True", provide an argument that an optimal schedule must contain the event with the earliest finish time (e.g., with a proof by contradiction as demonstrated by the examples in this week's discussion).

Solution: False.

Here's a counterexample. Consider the same events as above, but replace a's finish time with 1/2. The same schedule as before, b, d, c, e, is still optimal, but it doesn't contain the event with the earliest finish time.

Note that this is different from saying that *no* optimal solution contains the event with the earliest finish time. In fact, the solution given by the greedy algorithm *always* contains the event with the earliest finish time, but in general there may be other optimal solutions which do not.

Problem 2.

Consider the following problem: given an integer amount T, determine the minimum number of coins needed to make change for T cents using only quarters (25 cents), dimes (10 cents), nickels (5 cents), and pennies (1 cent).

For example, to make change for 55 cents, the minimum number of coins needed is three: two quarters and a nickel.

a) Describe a greedy algorithm for solving this problem. You may provide pseudocode or a description in words.

> **Solution:** Sort the coins in decreasing order of value. Then, starting with the largest coin, repeatedly take as many of that coin as possible until you can take no more. Then move on to the next largest coin, and so on.

b) What is the minimum number of coins (quarters, dimes, nickels and pennies) necessary to make change for 131 cents? List how many of each coin is needed.

> **Solution:** Using the greedy strategy, we take 5 quarters, a nickel, and a penny, for a total of 6 coins.

c) Suppose T is larger than 25 cents. True or False: the optimal solution must contain *exactly* $\lfloor T/25 \rfloor$ quarters.

If you answer "False", provide a counterexample in the form of a specific value of T and an optimal solution that does not contain exactly $\lfloor T/25 \rfloor$ quarters. If you answer "True", provide an argument that an optimal solution must contain exactly $\lfloor T/25 \rfloor$ quarters (e.g., using a proof by contradiction as demonstrated by the examples in this week's discussion).

> **Solution:** True.
>
> Suppose not – that is, suppose there is an optimal solution that uses fewer quarters (it can't possibly use more) but still adds up to T. Some subset of the coins in this solution must add up to 25 (since the total is greater than 25, and 25 is divisibly by 10, 5, and 1). This subset must contain at least 3 coins (the fewest number of coins required to make 25 cents without using a quarter). Exchanging these coins for a quarter gives a solution with fewer coins, but the same total, contradicting the assumption that the solution was optimal.
>
> Therefore, any optimal solution must contain exactly $\lfloor T/25 \rfloor$ quarters.

d) Now consider a more general version of the problem where coins have unspecified values $v_a > v_b > v_c > v_d$. It is **not** the case that the greedy algorithm will work here – it depends on what the coin values are.

Provide a set of coin values for which the greedy algorithm you described above will fail. Demonstrate that the greedy algorithm fails by writing down the solution it finds and showing that there is a better solution.

> **Solution:** Consider the coin values $v_a = 1, v_b = 3, v_c = 4, v_d = 5$. The greedy algorithm will fail for $T = 7$. It will take one of coin d (5 cents), then two of coin a (1 cent), for a total of 3 coins. But we can do better by taking one of coin c (4 cents) and one of coin b (3 cents), for a total of two coins.

Problem 3.

Consider the following problem. You are a mail carrier in a one-dimensional universe. You are given a collection of points, $\{p_1, p_2, \ldots, p_n\}$ on a line (i.e., points in $\mathbb{R}^1$) which represent houses you must deliver mail to.

a) Describe a greedy algorithm which finds a mail route (a sequence of houses to visit) which minimizes the total distance walked. You may start and end your route at any location (you do not need to account for getting to your starting location – you can assume that you're teleported there).

Solution: Sort the points in increasing order. Start with the smallest point. Then visit subsequence points in that order.

b) Suppose the houses are located at the points: (3, 13, 1, 7, 25, 10). Give an optimal route for this set of points.

Solution: Using our greedy algorithm, we get the route (1, 3, 7, 10, 13, 25).

c) Assume that you start your route at the leftmost point (e.g., in the example above, you start at point 1) and that the house locations are unique. True or False: in any optimal route where the first stop is the smallest point, the second stop must be at the second smallest point.

You do not need to provide justification for this part.

Solution: True.

Any route can be represented as a sequence of bars drawn on the number line from one stop to the next. In drawing the optimal route, you will start with your pencil on the leftmost point and draw a continuous line to the rightmost point. In this line, none of the segments between stops overlap. In physical terms, when you walk this route, you will never retrace your steps (i.e., you'll never walk the same ground twice).

Now, if you start at the leftmost house but walk past the second house to some other house, you will need to come back to the second house eventually. When you do, you'll retrace your steps, meaning that you'll walk the same ground twice. This cannot be optimal.